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 B.Sc. - Part - II (H), Paper - III, Date - 11-05-2020
 Differential Calculus (Problems based on
 Taylor's Theorem)

Q.1 In the equation

$$f(x+h) = f(x) + hf'(x) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) + \frac{h^n}{n!} f^{(n)}(x+\theta h)$$

if $f^{(n)}(x)$ is continuous, then prove that
 $\lim_{h \rightarrow 0} \theta = \frac{1}{n}$.

Soln. From the given eqn we have

$$f(x+h) = f(x) + hf'(x) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) + \frac{h^n}{n!} f^{(n)}(x+\theta h) \dots \text{--- (I)}$$

Since $f^{(n)}(x)$ is continuous, therefore, by Taylor's theorem $f(x+h)$

$$= f(x) + hf'(x) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) + \frac{h^n}{n!} f^{(n)}(x+\theta_1 h) \text{--- (II)}$$

where $0 < \theta_1 < 1$

From I and II, we have

$$\frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) + \frac{h^n}{n!} f^{(n)}(x+\theta_1 h) = \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x+\theta h)$$

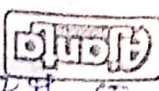
$$\Rightarrow f^{(n-1)}(x) + \frac{h}{n} f^{(n)}(x+\theta_1 h) = f^{(n-1)}(x+\theta h)$$

$$\Rightarrow \frac{h}{n} f^{(n)}(x+\theta_1 h) = f^{(n-1)}(x+\theta h) - f^{(n-1)}(x)$$

$$\Rightarrow \frac{1}{n} f^{(n)}(x+\theta_1 h) = \theta \cdot \frac{f^{(n-1)}(x+\theta h) - f^{(n-1)}(x)}{(x+\theta h) - (x)} \text{--- (III)}$$

By Lagrange's Mean value Theorem, we have

$$\frac{f^{(n-1)}(x+\theta h) - f^{(n-1)}(x)}{(x+\theta h) - (x)} = f^{(n)}(x+\theta_2 h), \text{ where } 0 < \theta_2 < 1$$

proof  B.Sc. (part-III), paper-III, 2014-15-16
 Rest part of Q.1.

∴ By III becomes

$$\frac{1}{h} f'(x+\theta_1 h) = \theta, f'(x+\theta_2 h)$$

Let $h \rightarrow 0$

$$\therefore \frac{1}{h} f'(x) = f'(x) \cdot \lim_{h \rightarrow 0} \theta$$

Hence, $\lim_{h \rightarrow 0} \theta = \frac{1}{h}$ proved

Q.2. If $f''(x)$ be continuous and not zero at $x=a$, show that $\lim_{h \rightarrow 0} \theta = \frac{1}{2}$.

Soln: We have, $f(x+h) = f(x) + h f'(x+\theta h)$.

Also, since $f''(x)$ is continuous,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x+\theta_1 h).$$

$$\therefore f(x) + h f'(x+\theta h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x+\theta_1 h)$$

$$\Rightarrow h f'(x+\theta h) = h \left[f'(x) + \frac{h}{2} f''(x+\theta_1 h) \right]$$

$$\Rightarrow f'(x+\theta h) - f'(x) = \frac{h}{2} f''(x+\theta_1 h)$$

$$\Rightarrow \frac{f'(x+\theta h) - f'(x)}{\theta h} = \frac{1}{2} \frac{1}{\theta} f''(x+\theta_1 h)$$

$$\Rightarrow f''(x+\theta_2 h) = \frac{1}{2} \frac{1}{\theta} f''(x+\theta_1 h), \text{ by mean-value theorem.}$$

$$\Rightarrow \theta f''(x+\theta_2 h) = \frac{1}{2} f''(x+\theta_1 h)$$

$$\text{Let } h \rightarrow 0. \text{ Then } f''(x) \lim_{h \rightarrow 0} \theta = \frac{1}{2} f''(x)$$

$$\Rightarrow \lim_{h \rightarrow 0} \theta = \frac{1}{2} \text{ Proved}$$