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B.Sc- part- II(H), paper- III, Date- 11-05-2020
Differential Calculus (Problems based on Taylor's Theorem)

Q.I In the equation

$$f(x+h) = f(x) + hf'(x) + \cdots + \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(x+\theta h)$$

if $f^n(x)$ is continuous, then prove that

$$\lim_{h\to 0} \theta = \frac{1}{n}.$$

Soln: From the given eqn we have

$$f(x+h) = f(x) + hf'(x) + \cdots + \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(x+\theta h) \cdots \text{(I)}$$

Since $f^n(x)$ is continuous, therefore, by Taylor's theorem $f(x+h)$

$$= f(x) + hf'(x) + \cdots + \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(x) + \frac{h^n}{\lfloor n} f^n(x+\theta_1 h) \quad \text{— II}$$

where $0 < \theta_1 < 1$

From I and II, we have

$$\frac{h^{n-1}}{\lfloor n-1} f^{n-1}(x) + \frac{h^n}{\lfloor n} f^n(x+\theta_1 h) = \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(x+\theta h)$$

$$\Rightarrow f^{n-1}(x) + \frac{h}{n} f^n(x+\theta_1 h) = f^{n-1}(x+\theta h)$$

$$\Rightarrow \frac{h}{n} f^n(x+\theta_1 h) = f^{n-1}(x+\theta h) - f^{n-1}(x)$$

$$\Rightarrow \frac{1}{n} f^n(x+\theta_1 h) = \theta \cdot \frac{f^{n-1}(x+\theta h) - f^{n-1}(x)}{(x+\theta h) - (x)} \quad \text{— III}$$

By Lagrange's Mean value Theorem, we have

$$\frac{f^{n-1}(x+\theta h) - f^{n-1}(x)}{(x+\theta h) - (x)} = f^n(x+\theta\theta_2 h), \text{ where } 0 < \theta_2 < 1$$

Rest part of Q. 1.

$\therefore$ By III becomes

$$\frac{1}{n} f^n(x+\theta_1 h) = \theta \cdot f^n(x+\theta\theta_2 h)$$

Let $h \to 0$

$$\therefore \frac{1}{n} f^n(x) = f^n(x) \cdot \lim_{h\to 0} \theta$$

Hence, $\lim_{h\to 0} \theta = \frac{1}{n}$ Proved.

Q.2. If $f''(x)$ be continuous and not zero at $x=a$, show that $\lim_{h\to 0} \theta = \frac{1}{2}$.

Soln: We have, $f(x+h) = f(x) + h f'(x+\theta h)$.

Also, since $f''(x)$ is continuous,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{\lfloor 2} f''(0+\theta_1 h).$$

$$\therefore f(x) + h f'(x+\theta h) = f(x) + h f'(x) + \frac{h^2}{\lfloor 2} f''(x+\theta_1 h)$$

$$\Rightarrow h f'(x+\theta h) = h\left[f'(x) + \frac{h}{\lfloor 2} f''(x)+\theta_1 h)\right]$$

$$\Rightarrow f'(x+\theta h) - f'(x) = \frac{h}{\lfloor 2} f''(x+\theta_1 h)$$

$$\Rightarrow \frac{f'(x+\theta h) - f'(x)}{\theta h} = \frac{1}{2}\frac{1}{\theta} f''(x+\theta_1 h)$$

$\Rightarrow f''(x+\theta_2\theta h) = \frac{1}{2}\frac{1}{\theta} f''(x+\theta_1 h)$, by mean-value theorem.

$$\Rightarrow \theta f''(x+\theta_2\theta h) = \frac{1}{2} f''(x+\theta_1 h)$$

Let $h \to 0$. Then $f''(x) \lim_{h\to 0} \theta = \frac{1}{2} f''(x)$

$$\Rightarrow \lim_{h\to 0} \theta = \frac{1}{2}$$ Proved.

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